

SMALLEST TETRAVALENT HALF-ARC-TRANSITIVE GRAPHS WITH THE VERTEX-STABILISER ISOMORPHIC TO THE DIHEDRAL GROUP OF ORDER 8

PRIMOŽ POTOČNIK AND ROK POŽAR

ABSTRACT. A connected graph whose automorphism group acts transitively on the edges and vertices, but not on the set of ordered pairs of adjacent vertices of the graph is called half-arc-transitive. It is well known that the valence of a half-arc-transitive graph is even and at least four. Several infinite families of half-arc-transitive graphs of valence four are known, however, in all except four of the known specimens, the vertex-stabiliser in the automorphism group is abelian. The first example of a half-arc-transitive graph of valence four and with a non-abelian vertex-stabiliser was described in [Conder and Marušič, *A tetravalent half-arc-transitive graph with non-abelian vertex stabilizer*, J. Combin. Theory Ser. B **88** (2003) 67–76]. This example has 10752 vertices and vertex-stabiliser isomorphic to the dihedral group of order 8. In this paper, we show that no such graphs of smaller order exist, thus answering a frequently asked question.

1. INTRODUCTION

Let Γ be a connected finite graph and G a group of automorphisms of Γ . If G acts transitively on the set of *vertices*, *edges* or *arcs* of the graph (an *arc* is an ordered pair of adjacent vertices), then Γ is said to be *G -vertex-transitive*, *G -edge-transitive* or *G -arc-transitive*, respectively. Furthermore, if G acts transitively on the vertices, edges, but not arcs of the graph, then Γ is *$(G, \frac{1}{2})$ -arc-transitive*. A graph Γ is *$\frac{1}{2}$ -arc-transitive* if it is $(G, \frac{1}{2})$ -arc-transitive for $G = \text{Aut}(\Gamma)$.

While graphs that are $(G, \frac{1}{2})$ -arc-transitive for some group of automorphisms G are rather easy to find (for example, every cycle is such a graph), $\frac{1}{2}$ -arc-transitive graphs are considerably more elusive and the question of their existence has been posed as an open problem by Tutte [28] and answered affirmatively by Bouwer [3] a few years later. Since cycles are arc-transitive and every $(G, \frac{1}{2})$ -arc-transitive graph has to have even valence (as observed already in [28]), the smallest admissible valence for a $\frac{1}{2}$ -arc-transitive graph is 4; and indeed, the smallest $\frac{1}{2}$ -arc-transitive graph is tetravalent and of order 27. The tetravalent $\frac{1}{2}$ -arc-transitive graphs (and half-arc-transitive graphs in general) were much studied by many authors from different points of view, ranging from purely combinatorial [7, 13, 17, 25, 26, 27, 29], geometrical [15], to permutation group theoretical [1, 8, 10] and abstract group theoretical [16, 21, 24].

2000 *Mathematics Subject Classification.* 20B25.

Key words and phrases. graph, tetravalent, vertex-transitive, edge-transitive.

Supported in part by Slovenian Research Agency, projects L1–4292, J1–5433, J1–6720, P1–0294 and P1–0285.

As observed by Marušič and Nedela [16], a group theoretical result of Glauberman [6] implies that the vertex-stabiliser G_v in a tetravalent $(G, \frac{1}{2})$ -arc-transitive graph is a group of order 2^s for some $s \geq 1$, of nilpotency class at most 2, generated by s involutions, and satisfying certain addition conditions (see [16, Theorem 1.1] for details and [21, Theorem 1.2] for a generalisation of this result to graphs of larger valence).

While each of the 2-groups described in [16, Theorem 1.1] can indeed occur as the vertex-stabiliser G_v in a tetravalent $(G, \frac{1}{2})$ -arc-transitive graph, it remains an open problem which of the groups of [16, Theorem 1.1] can occur as the vertex-stabiliser in the full automorphism group of a tetravalent $\frac{1}{2}$ -arc-transitive graph. This problem was resolved for the case of abelian vertex-stabilisers in [14], where for every positive integer s , a tetravalent $\frac{1}{2}$ -arc-transitive graph with G_v isomorphic to \mathbb{Z}_2^s was constructed.

Non-abelian vertex-stabilisers seem to be much more elusive in this respect. The first example of a tetravalent $\frac{1}{2}$ -arc-transitive graph with a non-abelian vertex-stabiliser has been constructed by Conder and Marušič [4]. Their example has order 10752 and vertex-stabiliser isomorphic to the dihedral group D_4 of order 8. In [5], two more examples with $\text{Aut}(\Gamma)_v \cong D_4$ were found (another one of order 10752 and one of order 21870), and also the first known example of a tetravalent $\frac{1}{2}$ -arc-transitive graph with a non-abelian vertex-stabiliser of order 16; the latter having order $90 \cdot 3^{10}$. To the best of our knowledge, these four graphs are the only known tetravalent $\frac{1}{2}$ -arc-transitive graphs with a non-abelian vertex-stabiliser.

Given that the order of the smallest tetravalent $\frac{1}{2}$ -arc-transitive graph with the stabiliser isomorphic to D_4 , found by Conder and Marušič more than a decade ago, is rather large, the question of existence of a smaller specimen of this family has often been raised by experts in this area. Perhaps surprisingly, as we prove in this paper, this question has a negative answer:

Theorem 1. *There are no tetravalent $\frac{1}{2}$ -arc-transitive graphs of order less than 10752 with the vertex-stabiliser isomorphic to D_4 , and there are precisely two such examples of order 10752.*

The proof of this theorem is a combination of a few theoretical results (see Lemma 6 and Lemma 7) relying heavily on a surprising and deep result of Praeger [19] (see Theorem 5), and computer assisted computations, based on the theory of lifting automorphisms along covering projection of graphs. Using this approach, we were able to produce a complete list of all tetravalent $(G, \frac{1}{2})$ -arc-transitive graphs of order at most 10752 with $G_v \cong D_4$. It transpires that there are precisely 564 such graphs (see Theorem 8). The MAGMA [2] code that generates these graphs can be found at [22]. Theorem 1 follows by inspection of the full automorphism groups of these 564 graphs – it turns out that all but two of them have automorphism group acting transitively on the arcs.

We thank Pablo Spiga and Gabriel Verret for reading the first draft of this paper and making some valuable comments.

2. PRELIMINARIES

2.1. Concerning graphs. Perhaps the most accepted definition of a graph is that of a *simple graph*, an object determined by its *vertex-set* V and its *edge-set* E satisfying $E \subseteq \{e : e \subseteq V, |e| = 2\}$. Even though we are mainly interested in

simple graph (especially since non-simple tetravalent edge-transitive graphs are easily classified; see Lemma 2), allowing graphs to have loops, multiple edges and semiedges proves to be most convenient in the proofs; in particular, a most useful Lemma 3 does not hold if one insists on allowing simple graphs only.

In what follows, we will therefore adopt a slightly more general model of a graph. As in, say, [12], for us, a *graph* will be an ordered 4-tuple $(D, V; \text{beg}, \text{inv})$ where D and $V \neq \emptyset$ are disjoint finite sets of *darts* and *vertices*, respectively, $\text{beg} : D \rightarrow V$ is a mapping which assigns to each dart x its *initial vertex* $\text{beg } x$, and $\text{inv} : D \rightarrow D$ is an involution which interchanges every dart x with its *inverse dart*, also denoted by x^{-1} . If Γ is a graph, then we let $D(\Gamma)$ and $V(\Gamma)$ denote its dart-set and its vertex-set, respectively.

The cardinality $|V|$ of V is called the *order* of Γ . The *neighbourhood* of a vertex v , denoted $\Gamma(v)$, is the set consisting of all the darts x with $\text{beg}(x) = v$, and the cardinality of $\Gamma(v)$ is called the *valence* of v . A graph is *tetravalent* if all of its vertices have valence 4.

The orbits of inv are called *edges*. The edge containing a dart x is called a *semiedge* if $\text{inv } x = x$, a *loop* if $\text{inv } x \neq x$ while $\text{beg}(x^{-1}) = \text{beg } x$, and is called a *link* if $\text{inv } x \neq x$ and $\text{beg}(x^{-1}) \neq \text{beg } x$. The *endvertices of an edge* are the initial vertices of the darts contained in the edge. Two links are *parallel* if they have the same endvertices.

A graph with no semiedges, no loops and no parallel links is called a *simple graph* and can be given uniquely in the usual manner, by its vertex-set and edge-set. Conversely, any simple graph, given in terms of its vertex-set V and edge-set E can be easily viewed as the graph $(D, V; \text{beg}, \text{inv})$, where $D = \{(u, v) \mid \{u, v\} \in E\}$, $\text{inv}(u, v) = (v, u)$ and $\text{beg}(u, v) = u$ for any $(u, v) \in D$.

Let $\Gamma = (D, V; \text{beg}, \text{inv})$ and $\Gamma' = (D', V'; \text{beg}', \text{inv}')$ be two graphs. A *morphism of graphs*, $f : \Gamma \rightarrow \Gamma'$, is a function $f : V \cup D \rightarrow V' \cup D'$ such that $f(V) \subseteq V'$, $f(D) \subseteq D'$, $f \circ \text{beg} = \text{beg}' \circ f$ and $f \circ \text{inv} = \text{inv}' \circ f$. A graph morphism is an *epimorphism* (*automorphism*) if it is a surjection (bijection, respectively). If g and h are automorphisms of Γ and x is a dart or a vertex of Γ , then we denote the g -image of x by x^g and multiply the automorphisms g and h so that $(v^{gh}) = (v^g)^h$. The set of automorphisms of Γ together with the above product forms a group, called the *automorphism group* of Γ and denoted by $\text{Aut}(\Gamma)$.

The graph Γ is called *vertex-transitive* (*dart-transitive*, respectively), provided that $\text{Aut}(\Gamma)$ acts transitively on vertices (darts, respectively) of X . Note that in the context of simple graphs, a *dart* is often called an *arc* of a graph; hence the term *arc-transitive* is also used as a synonym for dart-transitive.

An example of non-simple tetravalent edge-transitive (indeed, dart-transitive) graph is a *doubled cycle* $C_n^{(2)}$, which can be viewed as the usual cycle on n vertices, but with each edge doubled. Formally, the graph $C_n^{(2)}$ is $(V, D, \text{beg}, \text{inv})$, where $V = \mathbb{Z}_n$, $D = \mathbb{Z}_n \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\text{beg}(i, \epsilon, j) = i$, and $\text{inv}(i, \epsilon, j) = (i + (-1)^\epsilon, 1 - \epsilon, j)$ for every $i \in \mathbb{Z}_n$ and $\epsilon, j \in \mathbb{Z}_2$. If $n = 1$, then $C_n^{(2)}$ is the graph with a single vertex and two loops attached to it, while in the case $n = 2$, $C_n^{(2)}$ consists of two vertices and 4 parallel links connecting them. We state the following lemma without a proof:

Lemma 2. *A connected tetravalent edge-transitive graph is either simple, isomorphic to $C_n^{(2)}$ for some positive integer n , or isomorphic to the graph with a single vertex and four semiedges attached to it.*

2.2. Concerning quotients and covers. This section summarises some of the facts about quotients and covers that can be derived easily from [11] or [12].

An epimorphism $\wp: \Gamma \rightarrow \Lambda$ is a *covering projection* provided that the restriction $\wp_v: \Gamma(v) \rightarrow \Lambda(\wp(v))$ of \wp to the neighbourhood of v is bijective for every $v \in V(\Gamma)$. For simplicity, we shall also require both Γ and Λ to be connected. The preimage $\wp^{-1}(x)$ of a vertex or a dart x of Λ is called a *fibre* of the covering projection \wp and the group of all automorphisms of Γ that preserve each fibre set-wise is called the *group of covering transformations*. If the latter is transitive on each fibre, then it acts regularly on each fibre, and the covering projection is *regular*. A regular covering projection with the group of covering transformations being elementary abelian or solvable is itself called *elementary abelian* or *solvable*, respectively.

If $\wp: \Gamma \rightarrow \Lambda$ is a covering projection, $g \in \text{Aut}(\Gamma)$ and $h \in \text{Aut}(\Lambda)$ such that $\wp(x^g) = \wp(x)^h$ for every $x \in V(\Gamma) \cup D(\Gamma)$, then we say that g *projects* along \wp to h and that h *lifts* along \wp to g . A subgroup $H \leq \text{Aut}(\Lambda)$ *lifts* along \wp if and only if each of its elements lifts; in this case, the set of all the lifts of elements of H forms a group, called *the lift of H along \wp* . If H lifts along \wp , then we also say that \wp is *H -admissible*. Note that the group of covering transformations is nothing but the lift of the trivial group of automorphisms of Λ .

The composition $\wp_2 \circ \wp_1$ of two covering projections $\wp_1: \Gamma_1 \rightarrow \Gamma_2$ and $\wp_2: \Gamma_2 \rightarrow \Gamma_3$ is again a covering projection. If both \wp_1 and \wp_2 are regular, so is the composition $\wp_2 \circ \wp_1$. If a group $H \leq \text{Aut}(\Gamma_3)$ lifts along \wp_2 to some group that lifts further along \wp_1 , and if neither of \wp_1 and \wp_2 is a graph-isomorphism, then we say that the covering projection $\wp_2 \circ \wp_1$ *splits with respect to H* . An H -admissible covering projection that does not split with respect to H is said to be a *minimal H -admissible covering projection*.

Inspired by a paper of Lorimer [9] on arc-transitive graphs of prime valence, Praeger [18] introduced the concept of *normal quotients*, which has now become a standard tool in studying symmetries of graphs. Here we adapt this concept slightly so as to fit into the setting of graphs admitting loops, multiple edges and semiedges. This adaptation will prove most useful in the proof of our main result.

Let Γ be a graph and let $N \leq \text{Aut}(\Gamma)$. Let $D_N = \{x^N : x \in D(\Gamma)\}$ and $V_N = \{v^N : v \in V(\Gamma)\}$ denote the sets of N -orbits on the darts and vertices of Γ , respectively. Further, let $\text{beg}_N: D_N \rightarrow V_N$ and $\text{inv}_N: D_N \rightarrow V_N$ be defined by $\text{inv}_N(x^N) = \text{beg}(x)^N$ and $\text{inv}_N(x^N) = \text{inv}(x)^N$. The quadruple $\Gamma/N = (V_N, D_N, \text{beg}_N, \text{inv}_N)$ is then called the *quotient graph* of Γ with respect to N . Note that the mapping $\wp_N: \Gamma \rightarrow \Gamma/N$, defined by $\wp_N(x) = x^N$ for every $x \in V(\Gamma) \cup D(\Gamma)$, is a graph epimorphism, called the *normal quotient projection relative to N* .

If N is a normal subgroup of a group $G \leq \text{Aut}(\Gamma)$, then there is an obvious, but not necessarily faithful action of the quotient group G/N on the graph Γ/N . Note also that if G acts transitively on vertices, darts or edges of Γ , then so does G/N on Γ/N .

A natural question that arises at this point is under what circumstances the quotient projection is a covering projection. The following lemma provides two very simple (and easy-to-prove) sufficient and necessary conditions for this to happen. Note that the analogue result does not hold if one insists on considering simple graphs only and the concept of normal quotients as defined in [18].

Lemma 3. *Let Γ be a connected graph, let $N \leq \text{Aut}(\Gamma)$ and let $\wp: \Gamma \rightarrow \Gamma/N$ be the corresponding quotient projection. Then the following statements are equivalent:*

- (1) N is semiregular (that is, $N_v = \mathbf{1}$ for every $v \in V(\Gamma)$);
- (2) for every $v \in V(\Gamma)$, the valence of v in Γ equals the valence of $\wp(v)$ in Γ/N ;
- (3) \wp is a regular covering projection and N is the group of covering transformations of \wp .

We finish the section by presenting a basic but very useful result about regular covering projections.

Lemma 4. *Let Γ be a graph, let $G \leq \text{Aut}(\Gamma)$ and let N be a normal subgroup of G . If the quotient projection $\wp: \Gamma \rightarrow \Gamma/N$ is a covering projection, then the action of G/N on $V(\Gamma/N) \cup D(\Gamma/N)$ is faithful, and the stabilisers G_v and $(G/N)_{vN}$ are isomorphic for every $v \in V(\Gamma)$. Moreover, the group G projects to G/N and G is the lift of G/N along \wp . The group G is transitive on vertices, darts or edges of Γ if and only if G/N is transitive on vertices, darts or edges of Γ/N . In particular, Γ is $(G, \frac{1}{2})$ -arc-transitive if and only if Γ/N is $(G/N, \frac{1}{2})$ -arc-transitive.*

3. PROOF OF THE MAIN RESULT

We begin this section by pointing out an intimate relationship between tetravalent $(G, \frac{1}{2})$ -arc-transitive graphs and G -arc-transitive digraphs of valence 2. Let us first explain, how we understand the notion of a *digraph*.

Here, a *digraph* is a pair $\vec{\Gamma} = (\Gamma, \Delta)$, where Γ is a graph (called the *underlying graph* of $\vec{\Gamma}$) and $\Delta \subseteq D(\Gamma)$ such that $|\Delta \cap \{x, x^{-1}\}| = 1$ for every dart $x \in D(\Gamma)$; the set Δ is then called the *arc-set* of $\vec{\Gamma}$, and denoted $A(\vec{\Gamma})$. An *automorphism* of the digraph $\vec{\Gamma}$ is an automorphism of Γ that preserves Δ set-wise, and a group G of automorphisms of $\vec{\Gamma}$ is *arc-transitive* provided that G acts transitively on Δ . The digraph $\vec{\Gamma}$ is k -valent if every vertex is the initial vertex of precisely k darts in Δ as well as k darts in $D(\Gamma) \setminus \Delta$.

If Γ is a $(G, \frac{1}{2})$ -arc-transitive graph of valence $2k$, then G has two orbits on $D(\Gamma)$, each orbit containing precisely one dart of each edge. If Δ is one of these orbits, then (Γ, Δ) is a G -arc-transitive digraph of valence k ; we shall say that (Γ, Δ) *arises from* the $(G, \frac{1}{2})$ -arc-transitive graph Γ . Conversely, if $\vec{\Gamma}$ is a G -arc-transitive digraph of valence k , then its underlying graph is $(G, \frac{1}{2})$ -arc-transitive and of valence $2k$. In this sense, the study of tetravalent $(G, \frac{1}{2})$ -arc-transitive graphs is equivalent to the study of 2-valent G -arc-transitive digraphs.

The following theorem represents one of the main ingredients of the proof of our main result. Its proof in the case of simple graphs follows directly from an astonishing characterisation of G -arc-transitive digraphs of prime valency for which the group G contains a non-semiregular abelian normal subgroup, proved by Praeger in [19].

Theorem 5. *Let Γ be a connected tetravalent $(G, \frac{1}{2})$ -arc-transitive graph. If G contains an abelian normal subgroup N such that $N_v \neq \mathbf{1}$ for some vertex v , then G is solvable and G_v is elementary abelian.*

Proof. Let $\vec{\Gamma}$ be the G -arc-transitive 2-valent digraph arising from (Γ, G) .

Suppose first that Γ is not simple. Then by Lemma 2, $\Gamma \cong C_n^{(2)}$ for some $n \geq 1$ (note that the graphs with semiedges do not allow $\frac{1}{2}$ -arc-transitive groups of automorphisms). If $n = 1$, then $\text{Aut}(\vec{\Gamma}) = \text{Aut}(\vec{\Gamma})_v \cong \mathbb{Z}_2$. Similarly, if $n = 2$, then $\text{Aut}(\vec{\Gamma}) \cong D_4$ and $\text{Aut}(\vec{\Gamma})_v \cong \mathbb{Z}_2^2$. In both cases, the automorphism group is

solvable and the vertex-stabiliser is elementary abelian, as claimed. We may thus assume that $n \geq 3$. Now consider a pair $\{e, e'\}$ of parallel links of $C_n^{(2)}$ and the pair $\{x, x'\}$ of arcs of $\vec{\Gamma}$ lying on these two links. If x and x' have the same initial vertex, then this is the case for every parallel pair of links, implying that $\vec{\Gamma}$ can be viewed as obtained from the doubled cycle $C_n^{(2)}$ by orienting all the edges consistently in the clock-wise direction. Note that $\text{Aut}(\vec{\Gamma})_v \cong \mathbb{Z}_2^n$ and $\text{Aut}(\vec{\Gamma}) \cong \mathbb{Z}_2^n \rtimes C_n$ in this case, implying that G is solvable and G_v is elementary abelian, as claimed. On the other hand, if x and x' have distinct initial vertices, then it can be easily seen that $\text{Aut}(\vec{\Gamma})$ acts faithfully on $V(\vec{\Gamma})$, is isomorphic to D_n , and that $\text{Aut}(\vec{\Gamma})_v \cong \mathbb{Z}_2$.

We may thus assume that Γ is simple. By [19, Theorem 2.9] it follows that $\vec{\Gamma}$ is isomorphic to a certain digraph $C_n(2, t)$ (for some integer t), defined in [19, Definition 2.6]. However, by [19, Theorem 2.8(c)], the automorphism group of such a digraph is solvable and the vertex-stabiliser is elementary abelian, concluding the proof of the theorem. \square

Lemma 6. *If Γ is a tetravalent $(G, \frac{1}{2})$ -arc-transitive graph with the vertex-stabiliser G_v being non-abelian, then G is non-solvable.*

Proof. Suppose that the lemma is false and let Γ be a minimal counter-example, that is, Γ is a graph of smallest order admitting a solvable $\frac{1}{2}$ -arc-transitive group of automorphisms G with G_v non-abelian.

Let N be a minimal normal subgroup of G . Since G is solvable, N is elementary abelian. Since G_v is non-abelian, Theorem 5 implies that $N_v = 1$. But then, by Lemma 3 and Lemma 4, Γ/N is a tetravalent $(G/N, \frac{1}{2})$ -arc-transitive graph with G/N solvable and the vertex-stabiliser in G/N being isomorphic to G_v , and thus non-abelian. However, this contradicts the minimality of Γ , and thus proves the lemma. \square

The subgroup N of a finite group G generated by all solvable normal subgroups of G is itself solvable and normal in G and is called the *maximal solvable normal subgroup* of G . Note that G/N then contains no non-trivial solvable normal subgroups. Such groups are called *semisimple* (or also groups with a *trivial solvable radical*).

Lemma 7. *Let Γ be a tetravalent $(G, \frac{1}{2})$ -arc-transitive graph with G non-solvable and let N be the maximal solvable normal subgroup of G . Then the quotient projection $\Gamma \rightarrow \Gamma/N$ is a solvable regular covering projection, with Γ/N a simple $(G/N, \frac{1}{2})$ -arc-transitive graph, along which the semisimple group G/N lifts.*

Proof. Observe first that all we need to show is that N is semiregular. The fact that Γ/N is simple then follows from Lemma 2 and the fact that G/N is non-solvable, and the rest follows from Lemma 3 and Lemma 4.

Suppose now that the lemma is false and let Γ be a minimal counterexample, (in terms of $|V(\Gamma)|$). Let G be the corresponding non-solvable $\frac{1}{2}$ -arc-transitive group of automorphisms of Γ and N the maximal solvable normal subgroup of G such that $N_v \neq 1$.

Since N is solvable and non-trivial, the socle $S = \text{soc}(N)$ is non-trivial and abelian. Since S is characteristic in N , it is normal in G . Since G is non-solvable, Theorem 5 implies that $S_v = 1$. By Lemma 3, the quotient projection $\Gamma \rightarrow \Gamma/S$ is then a covering projection, and by Lemma 4, Γ/S is a $(G/S, \frac{1}{2})$ -arc-transitive

graph. Note that N/S is the maximal solvable subgroup of G/S . Since G/S is non-solvable, the minimality of Γ implies that N/S acts semiregularly on $V(\Gamma/S)$. But then, by Lemma 4, $N_v = 1$, contradicting our assumption on N . \square

3.1. Constructing a census of $(G, \frac{1}{2})$ -arc-transitive graphs with $G_v \cong D_4$. To shorten the text, we will call a pair (Γ, G) , where Γ is a connected tetravalent $(G, \frac{1}{2})$ -arc-transitive graph with the vertex-stabiliser isomorphic to the dihedral group D_4 , *relevant*. Observe that if (Γ, G) is relevant, then $|G| = |G_v| |V(\Gamma)| = 8|V(\Gamma)|$.

Lemmas 6 and 7 suggest a strategy for constructing all relevant pairs (Γ, G) with Γ of order at most M (for a given fixed integer M).

- (1) Find the set \mathcal{S} of all semisimple groups of order at most $8M$.
- (2) Find the set \mathcal{P}_0 of all relevant pairs (Γ, G) with G isomorphic to some group in \mathcal{S} .
- (3) For every relevant pair $(\Gamma, G) \in \mathcal{P}_0$, find all solvable regular covering projections $\varphi: \tilde{\Gamma} \rightarrow \Gamma$ with $|V(\tilde{\Gamma})| \leq M$ along which G lifts, and compute the lift \tilde{G} of G along φ . Let $\tilde{\mathcal{P}}$ denote the set of all pairs $(\tilde{\Gamma}, \tilde{G})$ obtained in this way.

The union of the sets \mathcal{P}_0 and $\tilde{\mathcal{P}}$, is then the set of all the relevant pairs (Γ, G) with $|V(\Gamma)| \leq M$. Let us now discuss each step of this approach.

Step (1) is easy if one has a database of all simple groups of order at most $8M$ available. Namely, if G is a semisimple group, its socle $\text{soc}(G)$ is isomorphic to a group $T_1^{\alpha_1} \times \dots \times T_k^{\alpha_k}$ for some pairwise non-isomorphic non-abelian simple groups T_i and positive integers α_i . Moreover, G acts by conjugation upon $\text{soc}(G)$ faithfully and thus embeds into $\text{Aut}(\text{soc}(G))$. In this sense, we have $\text{soc}(G) \leq G \leq \text{Aut}(\text{soc}(G))$.

For example, if $M = 10752$, then, using the database of simple groups in MAGMA, one can easily show that there are precisely 100 semisimple groups of order at most $8M$.

One of the possible approaches to Step (2) relies on the classification of the “universal groups” acting half-arc-transitively on an infinite tetravalent tree. These groups were determined in [16] (see also [21] and [20, Section 3.2 and Table 1]). In short, it follows from these results that every relevant pair arises from the tetravalent infinite tree T_4 and a group

$$U \cong \langle a, b, c, g \mid a^2, b^2, c^2, g^{-1}agb, g^{-1}bgc, (ab)^2, (bc)^2, (ac)^2b \rangle$$

acting $\frac{1}{2}$ -arc-transitively on T_4 . Here the subgroup $\langle a, b, c \rangle$, which is isomorphic to D_4 , corresponds to the stabiliser $U_{\tilde{v}}$ of a vertex $\tilde{v} \in V(T_4)$ and g to an automorphism of T_4 mapping \tilde{v} to a neighbour of \tilde{v} . (Note that up to conjugacy in $\text{Aut}(T_4)$, the above group is the unique $\frac{1}{2}$ -arc-transitive subgroup of $\text{Aut}(T_4)$ satisfying $U_{\tilde{v}} \cong D_4$.)

Now, for every relevant pair (Γ, G) , there exists an epimorphism $f: U \rightarrow G$ such that for some vertex $v \in V(\Gamma)$, we have $G_v = f(\langle a, b, c \rangle)$ and $f(g)$ mapping v to a neighbour of v . In particular, if we know what f is, then the graph Γ can be reconstructed as the coset graph $\text{Cos}(G, f(\langle a, b, c \rangle), f(g))$ (see [20, Section 3] for more information).

Step (2) thus amounts to finding all epimorphisms from U to each of the 100 semisimple groups of order at most $8 \cdot 10752$. This was easily done by the algorithms implemented in MAGMA (here the fact that U can be generated by two elements, a and g , is very helpful). The computations yield the set \mathcal{P}_0 of 16 relevant pairs

(Γ, G) with G semisimple. The properties of these pairs are summarized in Table 1.

TABLE 1. Properties of relevant pairs (Γ, G) of the set \mathcal{P}_0 .

ID	$ \Gamma $	$ \text{Aut}(\Gamma) $	$\text{soc}(G)$	$ G $
1	42	672	$\text{PSL}_2(7)$	336
2	90	2880	Alt_6	720
3	90	2880	Alt_6	720
4	306	4896	$\text{PSL}_2(17)$	2448
5	702	22464	$\text{PSL}_3(3)$	5616
6	756	12096	$\text{U}_3(3)$	6048
7	1404	44928	$\text{PSL}_3(3)$	11232
8	1518	24288	$\text{PSL}_2(23)$	12144
9	1860	29760	$\text{PSL}_2(31)$	14880
10	1950	62400	$\text{PSL}_2(25)$	15600
11	1950	62400	$\text{PSL}_2(25)$	15600
12	5040	80640	Alt_8	40320
13	6486	103776	$\text{PSL}_2(47)$	51888
14	7056	225792	$\text{PSL}_2(7) \times \text{PSL}_2(7)$	56448
15	7056	225792	$\text{PSL}_2(7) \times \text{PSL}_2(7)$	56448
16	8610	137760	$\text{PSL}_2(41)$	68880

Finally, Step (3) relies on computing solvable regular covering projections along which a given group lifts, up to a prescribed order M of the respective covering graphs. Such an algorithm has been developed and implemented in MAGMA by the second author; it is described in detail in [23], and is based on the theory of elementary abelian regular covering projections, presented in [12]. The essence of this algorithm relies on the fact that for every solvable G -admissible regular covering projection $\wp: \Gamma \rightarrow \Lambda$ there exists a sequence of elementary abelian regular covering projections $\wp_i: \Gamma_i \rightarrow \Gamma_{i-1}$ for $i \in \{1, \dots, k\}$, and a sequence of groups $G_i \leq \text{Aut}(\Gamma_i)$ for $i \in \{0, \dots, k\}$, such that the following hold:

- (1) $\wp = \wp_1 \circ \dots \circ \wp_k$ (in particular, $\Gamma_0 = \Lambda$ and $\Gamma_k = \Gamma$);
- (2) $G_0 = G$;
- (3) \wp_i is a minimal G_{i-1} -admissible covering projection and G_i is the lift of G_{i-1} along \wp_i for every $i \in \{1, \dots, k\}$.

It is now clear that Step (3) can be completed as follows: For each admissible pair $(\Gamma, G) \in \mathcal{P}_0$, find all minimal G -admissible elementary abelian covering projections $\wp: \tilde{\Gamma} \rightarrow \Gamma$, and the corresponding lifts \tilde{G} of G , satisfying the condition $|\text{V}(\tilde{\Gamma})| \leq M$. Denote the set of all thus obtained pairs $(\tilde{\Gamma}, \tilde{G})$ by \mathcal{P}_1 and think of it as forming the first level of the computations. It is obvious that only those admissible pairs $(\Gamma, G) \in \mathcal{P}_0$ had to be considered, for which $|\text{V}(\Gamma)| \leq \frac{M}{2}$. In particular, if $M = 10752$, then only the first 12 pairs from Table 1 have to be considered as base pairs. This procedure is then repeated recursively for $i = 1, 2, \dots$, by applying it to the i -th level \mathcal{P}_i (instead of \mathcal{P}_0) and yielding the set \mathcal{P}_{i+1} (instead of \mathcal{P}_1), until all the graphs in \mathcal{P}_i have order larger than $\frac{M}{2}$. The union of all \mathcal{P}_i , $i \geq 1$, then forms the set $\tilde{\mathcal{P}}$ defined in the description of Step (3).

If $M = 10752$, then this procedure terminates at level 8 and results in the set $\tilde{\mathcal{P}}$ of 760 relevant pairs. To give the reader an idea of the number of covers involved together with computation times, we present some statistics in Table 2.

These were all run on a 2.93 GHz Quad-Core Intel® Xeon® processor X7350 at the Faculty of Mathematics and Physics, University of Ljubljana. For each of the 12 pairs (Γ, G) from \mathcal{P}_0 satisfying $|\mathcal{V}(\Gamma)| \leq 5381$, each row of Table 2 displays the number of solvable regular covering projections obtained on each level during the computations and the computation time, respectively.

TABLE 2. The number of covers obtained on each level together with computation times.

ID	Lev. 1	Lev. 2	Lev. 3	Lev. 4	Lev. 5	Lev. 6	Lev. 7	Lev. 8	Time
1	56	90	75	34	15	7	2	1	62 hrs 27 mins
2	31	51	43	20	9	3	-	-	1 hr 54 mins
3	33	75	73	34	15	5	-	-	2 hrs 43 mins
4	11	13	7	2	1	-	-	-	31 mins
5	6	6	2	-	-	-	-	-	8 mins
6	6	5	2	-	-	-	-	-	6 mins
7	4	2	-	-	-	-	-	-	1 min 27 secs
8	4	2	-	-	-	-	-	-	1 min 26 secs
9	3	1	-	-	-	-	-	-	34 secs
10	3	1	-	-	-	-	-	-	38 secs
11	3	1	-	-	-	-	-	-	38 secs
12	3	-	-	-	-	-	-	-	1 min 2 secs

Putting all of this together, we obtain 776 relevant pairs with graph-order at most 10752. Furthermore, considering graphs up to isomorphism we have the following result.

Theorem 8. *There are precisely 564 pairwise non-isomorphic connected tetravalent graphs of order at most 10752 admitting a $\frac{1}{2}$ -arc-transitive group G with G_v isomorphic to D_4 .*

For each of these 564 graphs, the approach ensures existence of a $\frac{1}{2}$ -arc-transitive group of automorphisms G with $G_v \cong D_4$. However, the full automorphism group of these graphs is in all but two cases larger than this guaranteed G and is in fact dart-transitive. The two exceptional graphs, where the $\frac{1}{2}$ -arc-transitive group G equals the full automorphism group, have order 10752 and are precisely the two graphs constructed in [4] and [5]. This confirms Theorem 1.

The data about graphs of our census is available on-line at [22]. The package contains three files. The “Census-GHAT-10752-Gv8.mgm” file contains MAGMA code that generates the corresponding graphs. To load the contents of this file into MAGMA, the file “Census-GHAT-10752-Gv8.txt” (containing the list of neighbours of each graph) is needed. Additionally, the “Census-GHAT-10752-Gv8.csv” file is a “comma separated values” file representing a spreadsheet containing some pre-computed graph invariants. Each line of this file represents one of the graphs in the census, and has four fields as follows:

- (i) ID : the ID number of the graph;
- (ii) $|V|$: the order of the graph;
- (iii) $|A_v|$: the order of the vertex-stabiliser in the automorphism group of the graph;
- (iv) AT : this field contains “true” if the graph is arc-transitive and “false” otherwise.

REFERENCES

- [1] J. A. Al-Bar, A. N. Al-Kenani, N. M. Muthana, C. E. Praeger, Finite edge-transitive oriented graphs of valency four: A global approach, *Preprint 2014*.
- [2] W. Bosma, J. Cannon and C. Playoust, The MAGMA Algebra System I: The User Language, *J. Symbolic Comput.* **24** (1997), 235–265.
- [3] I. Z. Bouwer, Vertex and edge transitive, but not 1-transitive, graphs. *Canad. Math. Bull.* **13** (1970), 231–237.
- [4] M. D. E. Conder, D. Marušič, A tetravalent half-arc-transitive graph with non-abelian vertex stabilizer, *J. Combin. Theory Ser. B* **88** (2003) 67–76.
- [5] M. D. E. Conder, P. Potočnik, P. Šparl, Some recent discoveries about half-arc-transitive graphs, *Ars. Math. Contemp.* **8** (2015).
- [6] G. Glauberman, Isomorphic subgroups of finite p -subgroups, *Canad. J. Math.* **23** (1971), 983–1022.
- [7] A. Hujdurović, K. Kutnar and D. Marušič, Half-arc-transitive group actions with a small number of alternets, *J. Combin. Theory Ser. A* **124** (2014), 114–129.
- [8] C. H. Li, Z. P. Lu, D. Marušič, On primitive permutation groups with small suborbits and their orbital graphs, *J. Algebra* **279** (2004), 749–770.
- [9] P. Lorimer, Vertex-transitive graphs: Symmetric graphs of prime valency, *J. Graph Theory* **8** (1984), 55–68.
- [10] A. Malnič, D. Marušič, Constructing 4-valent $\frac{1}{2}$ -Transitive Graphs with a Nonsolvable Automorphism Group, *J. Combin. Theory Ser. B* **75** (1999), 46–55.
- [11] A. Malnič, R. Nedela, and M. Škoviera, Lifting Graph Automorphisms by Voltage Assignments, *European J. Combin.* **21** (2000), 927–947.
- [12] A. Malnič, D. Marušič, P. Potočnik, *Elementary abelian covers of graphs*, Journal of Algebraic Combinatorics **20** (2004), 71–97.
- [13] D. Marušič, Half-transitive group actions on finite graphs of valency 4, *J. Combin. Theory Ser. B* **73** (1998), 41–76.
- [14] D. Marušič, Quartic half-arc-transitive graphs with large vertex stabilizers, *Discrete Math.* **299** (2005), 180–193.
- [15] D. Marušič, R. Nedela, Maps and half-transitive graphs of valency 4, *European J. Combin.* **19** (1998), 345–354.
- [16] D. Marušič, R. Nedela, On the point stabilizers of transitive groups with non-self-paired suborbits of length 2, *J. Group Theory* **4** (2001), 19–43.
- [17] D. Marušič, C. E. Praeger, Tetravalent graphs admitting half-transitive group actions: alternating cycles, *J. Combin. Theory Ser. B* **75** (1999), 188–205.
- [18] C. E. Praeger, An O’Nan-Scott Theorem for finite quasiprimitive permutation groups, and an application to 2-arc transitive graphs, *J. London Math. Soc.(2)* **47** (1993), 227–239.
- [19] C. E. Praeger, Highly arc transitive digraphs, *European J. Combin.* **10** (1989), 281–292.
- [20] P. Potočnik, P. Spiga, G. Verret, A census of 4-valent half-arc-transitive graphs and arc-transitive digraphs of valence two, *Ars Math. Contemp.* **8** (2015).
- [21] P. Potočnik, G. Verret, On the vertex-stabiliser in arc-transitive digraphs, *J. Combin. Theory Ser. B* **100** (2010), 497–509.
- [22] P. Potočnik, R. Požar, Census of tetravalent $(G, \frac{1}{2})$ -arc-transitive graphs with the vertex-stabiliser isomorphic to the dihedral group of order 8, <http://osebje.famnit.upr.si/~rok.pozar>.
- [23] R. Požar, Some computational aspects of solvable regular covers of graphs, *J. Symbolic Comput.* (2014), doi: 10.1016/j.jsc.2014.09.023
- [24] P. Spiga, G. Verret, On the order of vertex-stabilisers in edge- and vertex-transitive graphs with local group $\mathbb{Z}_p \times \mathbb{Z}_p$ or $\mathbb{Z}_p \wr \mathbb{Z}_2$, arXiv.
- [25] M. Šajna, Half-transitivity of some metacirculants, *Discrete Math.* **185** (1998), 117–136.
- [26] P. Šparl, A classification of tightly attached half-arc-transitive graphs of valency 4, *J. Combin. Theory Ser. B* **98** (2008), 1076–1108.
- [27] P. Šparl, On the classification of quartic half-arc-transitive metacirculants, *Discrete Math.* **309** (2009), 2271–2283.
- [28] W. T. Tutte, *Connectivity in graphs*, University of Toronto Press, Toronto, 1966.
- [29] S. Wilson, Semitransitive graph, *J. Graph Theory* **45** (2004), 1–27.

PRIMOŽ POTOČNIK,
FACULTY OF MATHEMATICS AND PHYSICS, UNIVERSITY OF LJUBLJANA, SLOVENIA;
ALSO AFFILIATED WITH
IAM, UNIVERSITY OF PRIMORSKA, KOPER, SLOVENIA
AND
INSTITUTE OF MATHEMATICS, PHYSICS, AND MECHANICS, LJUBLJANA, SLOVENIA
E-mail address: `primoz.potocnik@fmf.uni-lj.si`

ROK POŽAR, FAMNIT,
UNIVERSITY OF PRIMORSKA, KOPER, SLOVENIA
E-mail address: `pozar.rok@gmail.com`